Second Order Linear - Overview

Definition: Linear $a_n(x)\,y^{(n)} + a_{n-1}(x)\,y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x)$

2^{nd} order: $y'' + p(x)y' + q(x)y = f(x)$

Definition: Homogeneous (RHS = 0)

$$y'' + p(x)y' + q(x)y = 0$$

Theorem: (n linearly independent solutions) Linear, $y = c_1y_1 + c_2y_2 + \cdots + c_ny_n$

Theorem: (Wronskian) $W = \det\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & & & \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix} = \begin{cases} 0, & \text{Linearly Dependent} \\ \neq 0, & \text{Lin. Indep.} \end{cases}$

Theorem: (IVP - n conditions)

Solve the system for $c_1, c_2, \ldots$

Second Order Linear – Constant Coefficients
– Homogeneous

Ex: $y'' - 5y' - 6y = 0$ $\quad y(0) = 7$, $\quad y'(0) = 7$

Try $y = e^{mx}$

$y' = m e^{mx}$

$y'' = m^2 e^{mx}$

$$e^{-mx}\left(m^2 e^{mx} - 5m e^{mx} - 6 e^{mx} = 0\right)$$

$m^2 - 5m - 6 = 0$ [Auxilliary Equation]

$(m-6)(m+1) = 0$

$m = 6 \qquad m = -1$

$y = e^{6x} \qquad y = e^{-1x}$

if lin. ind. ✓

$Y = c_1 Y_1 + c_2 Y_2$

$Y = c_1 e^{6x} + c_2 e^{-x}$

$y(0) = 7$

$7 = c_1 + c_2$

$y'(0) = 7$

$y' = 6c_1 e^{6x} - c_2 e^{-x}$

$7 = 6c_1 - c_2$

$7 = c_1 + c_2$

$7 = 6c_1 - c_2$

Matrix? RREF?

$14 = 7c_1 \rightarrow c_1 = 2$

$7 = 2 + c_2 \rightarrow c_2 = 5$

$y = 2e^{6x} + 5e^{-x}$

Inspiration: 1st order

$y' + Py = 0$

$\mu = e^{\int P\,dx} = e^{Px}$

$\underset{u}{e^{Px}}\,\underset{v'}{y'} + \underset{u'}{Pe^{Px}}\,\underset{v}{y} = 0$

$(e^{Px}y)' = 0$

$e^{Px}y = c$

$y = ce^{-Px}$

$$W = \begin{vmatrix} e^{6x} & e^{-1x} \\ 6e^{6x} & -1e^{-1x} \end{vmatrix}$$

$W = -e^{5x} - 6e^{5x}$

$W = -7e^{5x} \neq 0$

Second Order Linear - Constant Coefficients

- Homogeneous w/ repeated roots

Ex: $y'' - 6y' + 9y = 0$

$y = e^{mx} \rightarrow m^2 - 6m + 9 = 0$

$(m-3)(m-3) = 0$

$m = 3 \qquad m = 3$

$y_1 = e^{3x}, \qquad \cancel{y_2 = e^{3x}}$

$$W = \det\begin{pmatrix} e^{3x} & e^{3x} \\ 3e^{3x} & 3e^{3x} \end{pmatrix}$$

$W = 3e^{6x} - 3e^{6x} = 0$

Reduction of Order

$y = u \cdot y_1$

$y = u e^{3x}$

$y' = 3u e^{3x} + u' e^{3x}$

$y'' = 9u e^{3x} + 3u' e^{3x} + 3u' e^{3x} + u'' e^{3x}$

$= 9u e^{3x} + 6u' e^{3x} + u'' e^{3x}$

$y'' - 6y' + 9y = 0$

$\cancel{9ue^{3x}} + 6u'e^{3x} + u''e^{3x} - 6(\cancel{3ue^{3x}} + u'e^{3x}) + 9(\cancel{ue^{3x}}) = 0$

$u'' e^{3x} = 0$

$\int \quad u'' = 0$

$\int \quad u' = c_1$

$u = c_1 x + c_2$

$y = (c_1 x + c_2) e^{3x}$

$y = c_1 x e^{3x} + c_2 e^{3x}$

$\uparrow \qquad \uparrow$

$y_2 \qquad y_1$

Note: Repeated Roots (constant coeffs, Homogeneous)

$(m-9)^4$

$y_1 = e^{9x}$

$y_2 = x e^{9x}$

$y_3 = x^2 e^{9x}$

$y_4 = x^3 e^{9x}$

Second Order Linear - Constant Coefficients

- Homogeneous w/ non-real roots

Ex: $y'' - 6y' + 10y = 0$

$m^2 - 6m + 10 = 0$

$-1 \quad -1$

$m^2 - 6m + 9 = -1$

$(m-3)^2 = -1$

$m - 3 = \pm i$

$m = 3 \pm i$

$$\boxed{\begin{array}{l}\text{Euler's Formula (exp/trig)} \\ m = a \pm bi \\ \hookrightarrow y = c_1 e^{ax}\cos bx + c_2 e^{ax}\sin bx\end{array}}$$

$y = c_1 e^{3x}\cos x + c_2 e^{3x}\sin x$

Second Order Linear - Constant Coefficients

- Inhomogeneous, but nice RHS

Ex: $y'' - y' - 2y = e^{2x}$

Method 1 - Reduction of order

Find a solution to

$y'' - y' - 2y = 0$

& use reduction of order,

Method 2 - Annihilators

1) Find y_c = complementary solution

$m^2 - m - 2 = 0$

$(m-2)(m+1) = 0$

$m = 2 \quad m = -1 \longrightarrow y_c = c_1 e^{2x} + c_2 e^{-1x}$

$y' - 3y = 0$

$Dy - 3y = 0$

$(D-3)y = 0$ - operator notation

$(D-a)e^{ax} = De^{ax} - ae^{ax}$

$= ae^{ax} - ae^{ax}$

$= 0$

2) Identify the annihilator of RHS

$(D-2)\left(y'' - y' - 2y = e^{2x}\right)$

$(D-2)\;(D-2)(D+1)\,y = 0$

Monster Equation

Common Annihilators
$e^{ax} \to D - a$
$\sin bx, \cos bx \to D^2 + b^2$
$x^n \to D^{n+1}$
$x^n e^{ax}\cos bx \to \left((D-a)^2 + b^2\right)^{n+1}$

3) Solve Monster Equation

$(m-2)^2(m+1) = 0$

$y = c_1 e^{2x} + c_2 x e^{2x} + c_3 e^{-1x}$

$y = y_c + y_p$

4) Find y_p

- Only y_c will have c_1, c_2
- y_p has no unknown constants.

$y_p = c_2 x e^{2x}$

$y_p = A x e^{2x}$

$y_p' = A2xe^{2x} + Ae^{2x}$

$y_p'' = A4xe^{2x} + A2e^{2x} + A2e^{2x}$

$y'' - y' - 2y = e^{2x}$

$4Axe^{2x} + 4Ae^{2x} - \left(2Axe^{2x} + Ae^{2x}\right) - 2Axe^{2x} = e^{2x}$

$3Ae^{2x} = e^{2x}$

$3A = 1$

$A = 1/3$

$y_p = \frac{1}{3} x e^{2x}$

$y = y_c + y_p$

$y = c_1 e^{2x} + c_2 e^{-1x} + \frac{1}{3} x e^{2x}$

Second Order Linear – Constant Coefficients

– Inhomogeneous, but nice RHS

Ex: $y'' - y' - 2y = e^{2x}$

y_c: $m^2 - m - 2 = 0$

$(m-2)(m+1) = 0$

$m = 2 \quad m = -1$

$y_c = c_1 e^{2x} + c_2 e^{-1x}$

Reduction of Order?

$y = u e^{2x}$

$y' = 2ue^{2x} + u' e^{2x}$

$y'' = 4ue^{2x} + 4u'e^{2x} + u'' e^{2x}$

$$\overbrace{4\cancel{u}e^{2x} + 4u'e^{2x} + u''e^{2x}}^{y''} - \overbrace{(2\cancel{u}e^{2x} + u'e^{2x})}^{y'} - \overbrace{2(\cancel{u}e^{2x})}^{2y} = e^{2x}$$

$u''e^{2x} + 3u'e^{2x} = e^{2x}$

$u'' + 3u' = 1$ No u – Let $w = u'$

So $w' = u''$

$w' + 3w = 1$ Now, it's first order.

$e^{3x}w' + 3e^{3x}w = e^{3x}$ $\mu = e^{\int 3dx} = e^{3x}$

$\int (e^{3x}w)' = e^{3x}$

$e^{-3x}\left[e^{3x}w = \frac{1}{3}e^{3x} + c\right]$

$w = \frac{1}{3} + ce^{-3x}$

$\int u' = \frac{1}{3} + ce^{-3x}$

$u = \frac{1}{3}x + c_1 e^{-3x} + c_2$

$$y = \underbrace{\frac{1}{3}xe^{2x}}_{y_p} + \underbrace{c_1 e^{-x} + c_2 e^{2x}}_{y_c}$$

$y_1 = t$ Eqn.

is a solution

r.bakes@hvcc.edu

$y = u \cdot t$

hvcc.edu/~r.bakes